Some Realization of $gl_q(n)$-covariant Oscillator Algebra and $gl_q(n)$-covariant q-Virasoro Algebra with q a root of unity.

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Abstract

In this paper some realization of $gl_q(n)$-covariant oscillators is obtained when q is a root of unity. And the $gl_q(n)$-covariant q-Virasoro algebra is presented by using the $gl_q(n)$-covariant oscillators.

1 Introduction

Quantum groups or q-deformed Lie algebra implies some specific deformations of classical Lie algebras.

From a mathematical point of view, it is a non-commutative associative Hopf algebra. The structure and representation theory of quantum groups have been developed extensively by Jimbo [1] and Drinfeld [2].

The q-deformation of Heisenberg algebra was made by Arik and Coon [3], Macfarlane [4] and Biedenharn [5]. Recently there has been some interest in more general deformations involving an arbitrary real functions of weight generators and including q-deformed algebras as a special case [6-10].

In this paper we review the $gl_q(n)$ oscillator algebra [11,12] and its representation when q is a root of unity. In this case we obtain some realizations

of the algebra. We use these results to present 'new' extension of q-Virasoro algebra which I call the $gl_q(n)$-covariant q-Virasoro algebra.

2 $gl_q(n)$-Covariant Oscillator Algebra

$gl_q(n)$-covariant oscillator algebra is defined as [11,12]

$$\bar{a}_i\bar{a}_j = \sqrt{q}\bar{a}_j\bar{a}_i, \quad (i<j)$$

$$a_i a_j = \frac{1}{\sqrt{q}} a_j a_i, \quad (i<j)$$

$$a_i\bar{a}_j = \sqrt{q}\bar{a}_j a_i, \quad (i \neq j)$$

$$a_i\bar{a}_i = 1 + q\bar{a}_i a_i + (q-1)\Sigma_{k=i+1}^{n}\bar{a}_k a_k, \quad (i=1,2,\cdots,n-1)$$

$$a_n\bar{a}_n = 1 + q\bar{a}_n a_n,$$

$$[N_i, a_j] = -\delta_{ij}a_j, \quad [N_i,\bar{a}_j] = \delta_{ij}\bar{a}_j, \quad (i,j=1,2,\cdots,n) \tag{1}$$

Here N_i plays a role of number operator and $a_i(\bar{a}_i)$ plays a role of annihilation(creation) operator. From now on, we restrict our concern to the case that q is an N-th primitive root of unity, $q = e^{2\pi i/N}$. From the above algebra one can obtain the relation between the number operators and mode opeartors as follows

$$\bar{a}_i a_i = q^{\Sigma_{k=i+1}^{n} N_k}[N_i], \tag{2}$$

where $[x]$ is called a q-number and is defined as

$$[x] = \frac{q^x - 1}{q-1}.$$

Let us introduce the Fock space basis $|n_1, n_2, \cdots, n_n>$ for the number operators $N_1, N_2, \cdots, N_n$ satisfying

$$N_i|n_1,n_2,\cdots,n_n> = n_i|n_1,n_2,\cdots,n_n>, \quad (n_1,n_2,\cdots,n_n = 0,1,2\cdots) \tag{3}$$

Then we have the following representation

$$a_i|n_1, n_2, \cdots, n_n >= \sqrt{q^{\Sigma_{k=i+1}^n n_k}[n_i]}|n_1, \cdots, n_i - 1, \cdots, n_n >$$

$$\bar{a}_i|n_1, n_2, \cdots, n_n >= \sqrt{q^{\Sigma_{k=i+1}^n n_k}[n_i + 1]}|n_1, \cdots, n_i + 1, \cdots, n_n > . \qquad (4)$$

From the above representation we know that there exists the ground state $|0, 0, \cdots, 0 >$ satisfying $a_i|0, 0 >= 0$ for all $i = 1, 2, \cdots, n$. Thus the state $|n_1, n_2, \cdots, n_n >$ is obtatind by applying the creation operators to the ground state $|0, 0, \cdots, 0 >$

$$|n_1, n_2, \cdots, n_n >= \frac{\bar{a}_n^{n_n} \cdots \bar{a}_1^{n_1}}{\sqrt{[n_1]! \cdots [n_n]!}}|0, 0, \cdots, 0 > . \qquad (5)$$

If we introduce the scale operators as follows

$$Q_i = q^{N_i}, \quad (i = 1, 2, \cdots, n), \qquad (6)$$

we have from the algebra (1)

$$[a_i, \bar{a}_i] = Q_i Q_{i+1} \cdots Q_n. \qquad (7)$$

Acting the operators Q_i's on the basis $|n_1, n_2, \cdots, n_n >$ produces

$$Q_i|n_1, n_2, \cdots, n_n >= q^{n_i}|n_1, n_2, \cdots, n_n > . \qquad (8)$$

In this representation the following relations are true:

$$[a_i, (\bar{a}_i)^n] = q^{1-n}[n] Q_i \cdots Q_n (\bar{a}_i)^{n-1} \qquad (9)$$

We must notice that if q is a N-th primitive root of unity, the dimension of the representation space becomes finite. In this case we obtain the following operator identities;

$$a_i^N = 0 \quad (\bar{a}_i)^N = 1 \qquad (10)$$

3 First Realization

The q-deformation of the Bargmann-Fock reptesentation of $gl_q(n)$ oscillators is realized by going over the space of analytic function of n complex variables $z_1, \cdots, z_n$ such as $|z_i|^2 \leq (1-q)^{-1}$, where the operators $a_i, \bar{a}_i, N_i$ are defined

$$\begin{aligned}
\bar{a}_i &= T_{i+1}^{1/2} \cdots T_n^{1/2} z_i \\
a_i &= \frac{1}{z_i} T_{i+1}^{1/2} \cdots T_n^{1/2} \frac{1-T_i}{1-q} \\
N_i &= z_i = \frac{\partial}{\partial z_i}.
\end{aligned} \tag{11}$$

Here T_i is a scaling operator satisfying

$$T_i^{\alpha} f(z_1, \cdots, z_i, \cdots, z_n) = f(z_i, \cdots, q^{\alpha} z_i, \cdots, z_n)$$

and the relation between T_i and N_i is

$$T_i = q^{N_i}$$

It can be easily checked that (11) satisfy the $gl_q(n)$ covariant oscillator algebra (1).

In the space of analytic n-variable function $f(z_1, \cdots, z_n)$, there exists an inner product such as

$$(f,g) = \int (\Pi_{i=1}^n d_q^2 z_i) \mu(z_1, \cdots, z_n) \bar{f}(z_1, \cdots, z_n) g(z_1, \cdots, z_n) \tag{12}$$

where the measure function $\mu(z_1, \cdots, z_n)$is defined as

$$\mu(z_1, \cdots, z_n) = \frac{1}{\pi^n} \Pi_{i=1}^n (exp_q(q|z_i|^2))^{-1} \tag{13}$$

where q-exponential function is defined as

$$\exp_q(x) = \Sigma_{n=0}^{\infty} \frac{x^n}{[n]!}.$$

The q-exponential function satisfies the following recurrence relation

$$\exp_q(qx) = [1 - (1-q)x]\exp_q(x) \tag{14}$$

From the above definition of the inner product, one can obtain the orthonormal basis in the space of analytic n-variable function as follows

$$u_n(\bar{z}_1, \cdots, \bar{z}_n) = \frac{\Pi_{i=1}^n \bar{z}_i^{n_i}}{\sqrt{\Pi_{i=1}^n [n_i]!}} \tag{15}$$

Then we can easily see that

$$(u_n, u_m) = \delta_{nm} \tag{16}$$

where we used

$$\int_0^{1/1-q} x^n e_q(qx)^{-1} d_q x = [n]! \tag{17}$$

4 Second Realization

Let S_N be the discretized circle with points only at the positions of N-th primitive root of unity,

$$S_N = \{1, q, q^2, \cdots, q^{N-1}\}.$$

Then the basis function of $gl_q(n)$ covariant oscillator system becomes a n variable function $f(q^{n_1}, q^{n_2}, \cdots, q^{n_n})$ defined on the domain $S_N \otimes S_N \otimes \cdots \otimes S_N$ (n-copies). Now we define the action of the exponetials of the position and momentum operators

$$(h_i f)(q^{n_1}, \cdots, q^{n_i}, \cdots, q^{n_n}) = f(q^{n_1}, \cdots, q^{n_i - 1}, \cdots, q^{n_n}),$$

$$(g_i f)(q^{n_1}, \cdots, q^{n_i}, \cdots, q^{n_n}) = q^{n_i} f(q^{n_1}, \cdots, q^{n_i}, q^{n_{i+1}-1} \cdots, q^{n_n - 1})$$

where $n_i = 0, 1, \cdots, N-1$. When q is a N-th primitive root of unity , they satisfy

$$
\begin{aligned}
h_i g_i &= q g_i h_i, \\
g_i g_j &= q g_j g_i \quad (i < j), \\
g_i^N &= h_i^N = 1 \\
h_i g_j &= g_j h_i \quad (i \neq j) \\
h_i h_j &= h_j h_i
\end{aligned}
\tag{18}
$$

Then a_i and $\bar{a}_i$ operators in algebra (1) are realized as follows

$$
\begin{aligned}
\bar{a}_i &= g_i, \\
a_i &= g_i^{-1} \frac{h_{i+1}^2 \cdots h_n^2 (1-h_i)^2}{1-q}
\end{aligned}
\tag{19}
$$

It can be easily checked that the above realization obeys the $gl_q(n)$ covariant oscillator algebra. From the above realization we can easily see that the $\bar{a}_i$'s are idempotent while a_i's are nilpotent;

$$
\begin{aligned}
(\bar{a}_i)^N &= 1 \\
a_i^N &= 0
\end{aligned}
\tag{20}
$$

5 $gl_q(n)$-Covariant q-Virasoro Algebra

Finally we would like to to mention that the realization of the $gl_q(n)$ oscillator algebra may be used to construct the $gl_q(n)$-covariant q-Virasoro algebra.

Like the q-Virasoro generator [13], we introduce the $l_n^{(i)}$ as follows

$$
l_m^{(i)} = (\bar{a}_i)^{m+1} a_i \tag{21}
$$

which lead to the algebra

$$[l_r^{(i)}, l_m^{(i)}] = q^{-m}[m-r]Q_iQ_{i+1}\cdots Q_n l_{r+m}^{(i)}$$

$$l_r^{(i)} l_m^{(j)} = q^{\frac{1}{2}m(r+2)} l_m^{(j)} l_r^{(i)}, \quad (i<j) \tag{22}$$

If we redefine the operators through

$$L_m^{(i)} = Q_i^{-1}\cdots Q_n^{-1} l_m^{(i)} \tag{23}$$

we then have the following algebra

$$q^{m-r} L_r^{(i)} L_m^{(i)} - L_m^{(i)} L_r^{(i)} = [m-r] L_{r+m}^{(i)}$$

$$L_r^{(i)} L_m^{(j)} = q^{\frac{1}{2}mr} L_m^{(j)} L_r^{(i)}, \quad (i<j) \tag{24}$$

which goes to the n-copies of the ordinary Virasoro algebra in the $q \to 1$ limit. However, when q is a root of unity, it is something new although we do not know its meaning and physical application.

6 Concluding Remark

In this paper we studied the $gl_q(n)$-covariant multimode oscillator algebra and some of its realization when the deformation parameter q is an N-th primitive root of unity. In this realization we found that the q-creation operator is idempotent while the q-annihilation operator is nilpotent, so the representation space becomes finite dimensional. We used these results to construct the $gl_q(n)$-covariant q-Virasoro generator and to find the new kind of deformed Virasoro algebra which I call the $gl_q(n)$-covariant q-Virasoro algebra.

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